

HERMITIAN u -INVARIANTS OVER FUNCTION FIELDS OF p -ADIC CURVES

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ABSTRACT. Let p be an odd prime. Let F be the function field of a p -adic curve. Let A be a central simple algebra of period 2 over F with an involution σ . There are known upper bounds for the u -invariant of hermitian forms over (A, σ) . In this article we compute the exact values of the u -invariant of hermitian forms over (A, σ) .

CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Division algebras over $\mathcal{A}_i(2)$ -fields	7
4. Division algebras over semi-global fields	8
5. Tensor product of quaternions over arbitrary fields	9
References	10

1. INTRODUCTION

Let A be a central simple algebra over a field K . Let σ be an involution on A . Let $k = K^\sigma = \{x \in K \mid \sigma(x) = x\}$. Suppose $\text{char } k \neq 2$. Suppose $\varepsilon \in \{1, -1\}$. If V is a finitely generated right A -module and $h : V \times V \rightarrow A$ is an ε -hermitian space over (A, σ) , the *rank* of h is defined to be $\text{Rank}(h) = \frac{\dim_K(V)}{\deg(A) \text{ind}(A)}$. Let $\text{Herm}^\varepsilon(A, \sigma)$ denote the category of ε -hermitian spaces over (A, σ) . The Hermitian u -invariant [Mah05, 2.1] of (A, σ, ε) is defined to be:

$$u(A, \sigma, \varepsilon) = \sup\{n \mid \text{there exists an anisotropic } h \in \text{Herm}^\varepsilon(A, \sigma), \text{Rank}(h) = n.\}$$

Suppose that σ and τ are involutions on A . Mahmoudi has proved that [Mah05, 2.2] if σ and τ are of the same type, then $u(A, \sigma, \varepsilon) = u(A, \tau, \varepsilon)$; if σ is orthogonal and τ is symplectic, then $u(A, \sigma, \varepsilon) = u(A, \tau, -\varepsilon)$; if σ is unitary, then $u(A, \sigma, 1) = u(A, \sigma, -1)$. Thus we have only three types of Hermitian u -invariants [Mah05, 2.3], we denote:

$$u(A, \sigma, \varepsilon) = \begin{cases} u^+(A), & \text{if } \varepsilon = 1 \text{ and } \sigma \text{ is orthogonal,} \\ & \text{or, } \varepsilon = -1 \text{ and } \sigma \text{ is symplectic;} \\ u^-(A), & \text{if } \varepsilon = -1 \text{ and } \sigma \text{ is orthogonal,} \\ & \text{or, } \varepsilon = 1 \text{ and } \sigma \text{ is symplectic;} \\ u^0(A), & \text{if } \sigma \text{ is unitary.} \end{cases}$$

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where u^+ is called the *orthogonal* Hermitian u-invariant, u^- is called the *symplectic* Hermitian u-invariant and u^0 is called the *unitary* Hermitian u-invariant.

In section 3, we provide upper bounds for Hermitian u-invariants of division algebras with Springer's property over $\mathcal{A}_i(2)$ -fields. For definitions of $\mathcal{A}_i(2)$ -fields and Springer's property, see the beginning of section 3.

Theorem 1.1. *Let D be a division algebra over a field K with an involution σ . Suppose $k = K^\sigma$, $\text{char } k \neq 2$, $\varepsilon \in \{1, -1\}$ and $d = \deg(D)$. Suppose k is an $\mathcal{A}_i(2)$ -field and D satisfies the Springer's property.*

- (i) *If σ is of the first kind, then $u^+(D) \leq (1 + \frac{1}{d})2^{i-1}$ and $u^-(D) \leq (1 - \frac{1}{d})2^{i-1}$;*
- (ii) *If σ is of the second kind, then $u^0(D) \leq 2^{i-1}$.*

Let p be an odd prime number. Let F be the function field of a smooth projective geometrically integral curve over a p -adic field. The field F is also called a *semi-global* field. Let D be a central division F -algebra with an involution σ of the first kind. Suppose $D \neq F$. As a consequence of an inequality of Mahmoudi [Mah05, 3.6] with $u(F) = 8$ ([PS10] or [HB10] and [Lee13]), $u^+(D) \leq 27$ and $u^-(D) \leq 10$. Parihar and Suresh [PS13] have proved that $u^+(D) \leq 14$ and $u^-(D) \leq 8$. We obtain exact values of Hermitian u-invariants:

Theorem 1.2. *Let F be the function field of a curve over a p -adic field with $p \neq 2$. Let D be a central division algebra over F . Let L/F be a quadratic extension.*

- (1) *if D is quaternion, then $u^+(D) = 6$ and $u^-(D) = 2$;*
- (2) *if D is quaternion and $D \otimes_F L$ is division, then $u^0(D \otimes_F L) = 4$;*
- (3) *if D is biquaternion, then $u^+(D) = 5$ and $u^-(D) = 3$.*

Let A be a central simple algebra over a field k . Suppose $\text{char } k \neq 2$ and $\text{per}(A) = 2$. Then, by a special case [Mer81] of the Merkurjev-Suslin theorem [MS82], A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for some quaternion algebras H_1, \dots, H_n over k . Let K/k be a quadratic extension. In [PS13], upper bounds for $u^+(A)$, $u^-(A)$, $u^0(A \otimes K)$ are given and they depend only on $u(k)$ and n . In section 5, we obtain sharper upper bounds for these Hermitian u-invariants. In fact we prove the following

Theorem 1.3. *Let A be a central simple algebra over a field k . Suppose $\text{char } k \neq 2$ and $\text{per}(A) = 2$. Suppose A is Brauer equivalent to $H_1 \otimes \cdots \otimes H_n$ for n quaternion algebras H_1, \dots, H_n over k .*

- (1) $u^+(A) \leq (\frac{4}{5} + \frac{1}{5}(\frac{9}{4})^n)u(k)$;
- (2) $u^-(A) \leq (-\frac{1}{5} + \frac{1}{5}(\frac{9}{4})^n)u(k)$;
- (3) $u^0(A \otimes_k K) \leq (\frac{1}{5} + \frac{3}{10}(\frac{9}{4})^n)u(k)$ for all quadratic extension K/k .

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2. PRELIMINARIES

Let K be a field. Let A be a central simple algebra over K with an involution σ . Let $k = K^\sigma$. We suppose $\text{char}(k) \neq 2$ throughout the paper. Let V be a finitely generated right A -module and $\varepsilon \in \{1, -1\}$. A map $h : V \times V \rightarrow A$ is called an ε -hermitian form over (A, σ) if h is bi-additive; $h(xa, yb) = \sigma(a)h(x, y)b$

for all $a, b \in A$, $x, y \in V$; and $h(y, x) = \varepsilon \sigma(h(x, y))$ for all $x, y \in A$. We call h an ε -hermitian space if given $h(x, y) = 0$ for all $x \in V$, we have $y = 0$. We say that h is *isotropic* if there exists $x \in V$, $x \neq 0$ such that $h(x, x) = 0$; otherwise we say that h is *anisotropic*.

Lemma 2.1 (Morita Invariance). *Let K , A , σ , k be as before. Suppose $A \simeq M_m(D)$ for a central division algebra D over K . Suppose σ is an involution on A and $\varepsilon \in \{1, -1\}$. Then there exists an involution τ on D and $\varepsilon_0 \in \{1, -1\}$ such that $u(A, \sigma, \varepsilon) = u(D, \tau, \varepsilon\varepsilon_0)$.*

Furthermore, $u^+(A) = u^+(D)$, $u^-(A) = u^-(D)$ and $u^0(A) = u^0(D)$.

Proof. It is a consequence of [Knu91, ch. I, 9.3.5] and [KMRT98, 4.2]. \square

From now on, we mostly focus on central division algebras.

Lemma 2.2. *Let D be a central division algebra over a field K with an involution σ . Let $k = K^\sigma$, $\text{char } k \neq 2$. Suppose k is a non-archimedean local field.*

- (1) *If σ is of the first kind and $D \neq k$, then $u^+(D) = 3$, $u^-(D) = 1$.*
- (2) *If σ is of the second kind, then $u^0(D) = 2$.*

Proof. See [Tsu61, Thm. 1, Thm. 3] and [Sch85, ch. 10, 2.2]. \square

We fix the following notations from 2.3 to 2.9. Let (k, v) be a complete discrete valued field with residue field \bar{k} , $\text{char } \bar{k} \neq 2$. Let D be a finite-dimensional division k -algebra with center K with an involution σ such that $K^\sigma = k$. By [CF67, ch. II, 10.1], v extends to a valuation v' on K such that $v'(x) = \frac{1}{[K:k]}v(N_{K/k}(x))$ for all $x \in K^*$. By [Wad86], v' extends to a valuation w on D such that $w(x) = \frac{1}{\text{ind}(D)}v'(\text{Nrd}_{D/K}(x))$ for all $x \in D^*$. Since $\text{Nrd}_{D/K}(x) = \text{Nrd}_{D/K}(\sigma(x))$, we have $w(\sigma(x)) = w(x)$ for all $x \in D$. Let $R_w = \{x \in D \mid w(x) \geq 0\}$ and $\mathfrak{m}_w = \{x \in D \mid w(x) > 0\}$. Let $\overline{D} = R_w/\mathfrak{m}_w$ be the residue division algebra (see [Rei03, 13.2]) of (D, w) over \bar{k} with involution $\overline{\sigma}$ such that $\overline{\sigma}(\overline{x}) = \overline{\sigma(x)}$ for all $x \in R_w$, where $\overline{x} = x + \mathfrak{m}_w$. Let $h: V \times V \rightarrow D$ be an ε -hermitian space over (D, σ) . By [Knu91, Ch. I, 6.2], V is free with an orthogonal basis $\{e_1, \dots, e_n\}$ such that $h(e_i, e_i) = a_i$ for some $a_i \in D$ with $\sigma(a_i) = \varepsilon a_i$ for all $1 \leq i \leq n$; and $h(e_i, e_j) = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq n$ and $i \neq j$. We denote $h = \langle a_1, \dots, a_n \rangle$. If $w(a_i) = 0$ for all $1 \leq i \leq n$, then $\overline{h} = \langle \overline{a}_1, \dots, \overline{a}_n \rangle \in \text{Herm}^\varepsilon(\overline{D}, \overline{\sigma})$. Let t_D be a parameter of (D, w) . By [Lar99, 2.7], there exists $\pi_D \in D$ such that $w(\pi_D) \equiv w(t_D) \pmod{2w(D^*)}$ and $\sigma(\pi_D) = \varepsilon' \pi_D$ for some $\varepsilon' \in \{1, -1\}$. Larmour proved the following hermitian analogue of a theorem of Springer.

Proposition 2.3 ([Lar06, 3.4] or [Lar99, 3.27]). *Let k , v , D , K , σ , w , h , π_D and ε' be as above. There exist $h_1 \in \text{Herm}^\varepsilon(D, \sigma)$, $h_2 \in \text{Herm}^{\varepsilon\varepsilon'}(D, \text{Int}(\pi_D) \circ \sigma)$, with $h \simeq h_1 \perp h_2 \pi_D$ and each diagonal entries of h_1 and h_2 have w -value 0. Further, the following are equivalent: (i) h is isotropic; (ii) h_1 or h_2 is isotropic; (iii) \overline{h}_1 or \overline{h}_2 is isotropic.*

Corollary 2.4. $u(D, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\text{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$.

Proof. Suppose $h \in \text{Herm}^\varepsilon(D, \sigma)$ and $h \simeq h_1 \perp h_2 \pi_D$ as in Proposition 2.3. Since $\text{Rank}(h) = \text{Rank}(h_1) + \text{Rank}(h_2) = \text{Rank}(\overline{h}_1) + \text{Rank}(\overline{h}_2)$, if $\text{Rank}(h) > u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\text{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$, then $\text{Rank}(\overline{h}_1) > u(\overline{D}, \overline{\sigma}, \varepsilon)$ or $\text{Rank}(\overline{h}_2) > u(\overline{D}, \overline{\text{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$. Then \overline{h}_1 or \overline{h}_2 is isotropic. By Proposition 2.3, h is isotropic. Hence $u(D, \sigma, \varepsilon) \leq u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\text{Int}(\pi_D) \circ \sigma}, \varepsilon\varepsilon')$.

Conversely, suppose $g_1 = \langle a_1, \dots, a_m \rangle \in \text{Herm}^\varepsilon(\overline{D}, \overline{\sigma})$ such that $\overline{\sigma}(a_i) = \varepsilon a_i$, $m = u(\overline{D}, \overline{\sigma}, \varepsilon)$ and g_1 is anisotropic. Since $a_i \neq 0$, there exists $b_i \in R_w$, $w(b_i) = 0$ such that $\overline{b_i} = a_i$. Let $c_i = \frac{1}{2}(b_i + \varepsilon \sigma(b_i))$. Then $\sigma(c_i) = \varepsilon c_i$ and $\overline{c_i} = a_i$. Let $h_1 = \langle c_1, \dots, c_m \rangle \in \text{Herm}^\varepsilon(D, \sigma)$. Then $\overline{h_1} = g_1$ and by [Lar06, 2.3], h_1 is anisotropic.

Suppose $g_2 = \langle a_{m+1}, \dots, a_{m+n} \rangle \in \text{Herm}^{\varepsilon'}(\overline{D}, \overline{\text{Int}(\pi_D) \circ \sigma})$ such that g_2 is anisotropic. Similar to the previous paragraph, there exists $h_2 \in \text{Herm}^{\varepsilon'}(D, \text{Int}(\pi_D) \circ \sigma)$ such that $\overline{h_2} = g_2$ and h_2 is anisotropic.

By Proposition 2.3, $h = h_1 \perp h_2 \pi_D$ is anisotropic and $\text{Rank}(h) = m + n$. $u(D, \sigma, \varepsilon) \geq u(\overline{D}, \overline{\sigma}, \varepsilon) + u(\overline{D}, \overline{\text{Int}(\pi_D) \circ \sigma}, \varepsilon')$. \square

Lemma 2.5. *Suppose D is ramified at the discrete valuation of k , $Z(D) = k$ and $\text{per}(D)|2$. Then there exist an involution σ on D of first kind and elements $\alpha, \pi_D \in D$ satisfying the following conditions:*

- (a) $\overline{\sigma}$ is an involution of the second kind;
- (b) α^2 is a unit at the valuation v of k and $Z(\overline{D}) = \overline{k}(\overline{\alpha})$;
- (c) π_D is a parameter of D , $\sigma(\pi_D) = \pm \pi_D$ and $\overline{\text{Int}(\pi_D) \circ \sigma}$ is of the first kind.

Proof. Suppose D is ramified. Then D is Brauer equivalent to $D_0 \otimes (u, \pi)$ with D_0 a central division algebra over k unramified at v , $\pi \in k^*$ a parameter and $u \in k^* \setminus k^{*2}$ a unit at v . Furthermore, \overline{D} Brauer equivalent to $\overline{D}_0 \otimes \overline{k}(\sqrt{u})$ and $Z(\overline{D}) = \overline{k}(\sqrt{u})$ [TW15, 8.77].

(a) By [CDT+95, prop. 4], the nontrivial automorphism of $Z(\overline{D})/\overline{k}$ extends to an involution on \overline{D} of the second kind and it can be lifted to an involution σ on D of the first kind.

(b) Since k is complete, by [CDT+95, p. 53, Lem. 1], there exists $\alpha \in D$ such that $\alpha^2 \in Z(D)$, $\overline{\alpha} \in Z(\overline{D})$ corresponds \sqrt{u} in the isomorphism $Z(\overline{D}) = \overline{k}(\sqrt{u})$ and $\sigma(\alpha) = -\alpha$.

(c) By [JW90, prop. 1.7], there exists a parameter $t_D \in D$ such that $\overline{\text{Int}(t_D)}$ is the non-trivial $Z(\overline{D})/\overline{k}$ -automorphism. Since $\overline{\sigma}$ is of the second kind and $\overline{\text{Int}(t_D)}$ induces the non-trivial automorphisms of $Z(\overline{D})$, $\overline{\text{Int}(t_D) \circ \sigma}$ is of the first kind. Since σ is an involution, $w(t_D) = w(\sigma(t_D))$ and hence $\overline{\sigma(t_D)t_D^{-1}} \neq 0 \in \overline{D}$.

Case 1: Suppose that $\overline{\sigma(t_D)t_D^{-1}} = 1$. Let $\pi_D = t_D + \sigma(t_D)$. Then $\sigma(\pi_D) = \pi_D$. Since $\pi_D t_D^{-1} = 1 + \sigma(t_D)t_D^{-1}$, $\pi_D t_D^{-1} = 1 + \overline{\sigma(t_D)t_D^{-1}} = 1 + 1 = 2 \neq 0$. Hence $w(\pi_D) = w(t_D)$. Since $\pi_D t_D^{-1} = 2$, $\overline{\text{Int}(\pi_D) \circ \sigma} = \overline{\text{Int}(t_D) \circ \sigma}$ and hence $\overline{\text{Int}(\pi_D) \circ \sigma}$ is of the first kind. Thus π_D satisfies (c).

Case 2: Suppose that $\overline{\sigma(t_D)t_D^{-1}} \neq 1$. Let $\pi_D = \alpha t_D - \sigma(\alpha t_D)$. Then $\sigma(\pi_D) = -\pi_D$. We have $\pi_D t_D^{-1} = \alpha - \sigma(t_D)\sigma(\alpha)t_D^{-1}$. Since $\overline{\sigma(\alpha)} = -\overline{\alpha}$ and $\overline{t_D \alpha t_D^{-1}} = -\overline{\alpha}$, we have

$$\begin{aligned} \overline{\pi_D t_D^{-1}} &= \overline{\alpha - \sigma(t_D)\sigma(\alpha)t_D^{-1}} \\ &= \overline{\alpha + \sigma(t_D)t_D^{-1}t_D \alpha t_D^{-1}} \\ &= \overline{\alpha + \sigma(t_D)t_D^{-1}(-\overline{\alpha})} \\ &= (1 - \overline{\sigma(t_D)t_D^{-1}})\overline{\alpha} \neq 0 \end{aligned}$$

Hence $w(\pi_D) = w(t_D)$. Since $\overline{\sigma(\alpha)} = -\overline{\alpha}$, $\alpha^2 \in k$ and $\overline{t_D \alpha t_D^{-1}} = -\overline{\alpha}$, we have $\overline{\sigma(t_D)\alpha\sigma(t_D)^{-1}} = -\overline{\alpha}$ and

$$\begin{aligned}
& \overline{(\pi_D \alpha \pi_D^{-1} + \alpha) \pi_D t_D^{-1}} \\
&= \overline{\pi_D \alpha t_D^{-1} + \alpha \pi_D t_D^{-1}} \\
&= \overline{(\alpha t_D - \sigma(t_D) \sigma(\alpha)) \alpha t_D^{-1} + \alpha (\alpha t_D - \sigma(t_D) \sigma(\alpha)) t_D^{-1}} \\
&= \overline{\alpha t_D \alpha t_D^{-1} - \sigma(t_D) \sigma(\alpha) \alpha t_D^{-1} + \alpha^2 + \alpha \sigma(t_D) \alpha t_D^{-1}} \\
&= \overline{-\alpha^2 + \sigma(t_D) \alpha^2 t_D^{-1} + \alpha^2 + \alpha (\sigma(t_D) \alpha \sigma(t_D)^{-1}) \sigma(t_D) t_D^{-1}} \\
&= \overline{\alpha^2 \sigma(t_D) t_D^{-1} - \alpha^2 \sigma(t_D) t_D^{-1}} = 0.
\end{aligned}$$

Since $\overline{\pi_D t_D^{-1}} \neq 0$, $\overline{\pi_D \alpha \pi_D^{-1} + \alpha} = 0$ and hence $\overline{(\text{Int}(\pi_D) \circ \sigma)(\bar{\alpha})} = \bar{\alpha}$. Thus π_D satisfies (c).

To summarize, σ , α and π_D satisfy required properties. \square

Corollary 2.6. *Suppose $Z(D) = k$ and $\text{per}(D) = 2$.*

(1) *If D is unramified at the discrete valuation of k , then*

$$u^+(D) = 2u^+(\bar{D}) \text{ and } u^-(D) = 2u^-(\bar{D}).$$

(2) *If D is ramified at the discrete valuation of k , then*

$$u^+(D) = u^0(\bar{D}) + u^+(\bar{D}) \text{ and } u^-(D) = u^0(\bar{D}) + u^-(\bar{D}).$$

Proof. Suppose D is unramified. Then we can take $\pi_D = \pi$, where π is a parameter of k . Since $\sigma(\pi) = \pi$, we have $\varepsilon' = 1$ and $\text{Int}(\pi_D) \circ \sigma = \sigma$. Hence, by Corollary 2.4, we have the required result.

Suppose D is ramified. Then choose σ and π_D as in Lemma 2.5. Then, by Corollary 2.4, we have the required result. \square

Let K/k be a quadratic extension. Let D be a central division algebra over k with an involution σ of the first kind. Then $\sigma \otimes \iota$ is an involution on $D \otimes_k K$ of the second kind with ι being the non-trivial automorphism of K/k .

Remark 2.7. Suppose $D \otimes K$ is division. Then there are three possibilities of ramification:

- (1) K/k is unramified and $D \otimes K/K$ is unramified;
- (2) K/k is unramified and $D \otimes K/K$ is ramified;
- (3) K/k is ramified and $D \otimes K/K$ is unramified;

We show that “ K/k is ramified and $D \otimes K/K$ is ramified” cannot happen.

In fact, if K/k is ramified, then $K = k(\sqrt{\pi})$ for some parameter $\pi \in k$ and $\bar{K} = \bar{k}$. If D/k is unramified, then $D \otimes K/K$ is unramified. Suppose D/k is ramified. Then D is Brauer equivalent to $D_0 \otimes (u, \pi)$ for some D_0 unramified on k and $u \in k$ a unit at the valuation of k [TW15, 8.77]. Thus $D \otimes K$ is Brauer equivalent to $D_0 \otimes K$ and hence $D \otimes K/K$ is unramified.

Consequently, (2) and (3) can be shortened as

- (2) $D \otimes K/K$ is ramified. (3) K/k is ramified.

Remark 2.8. Suppose we are in case (2) of Remark 2.7. Suppose $K = k(\sqrt{\lambda})$ and D is Brauer equivalent to $D_0 \otimes (u, \pi)$ for some D_0 unramified on k and $u \in k$ a unit at the valuation of k . Then $\bar{K} = \bar{k}(\sqrt{\lambda})$, $Z(\bar{D}) = \bar{k}(\sqrt{u})$ and $Z(\bar{D} \otimes \bar{K}) = \bar{k}(\sqrt{u}, \sqrt{\lambda})$. Here u and λ are in different square classes of k , otherwise $(u, \pi)_K$ is split and hence $D \otimes K$ is unramified over K . Since $\bar{D} \otimes \bar{K} = \bar{D} \otimes \bar{k}(\sqrt{u}, \sqrt{\lambda})$ and \bar{D} has an involution of the first kind, $\bar{D} \otimes \bar{K}$ has three possible types of involutions of second kind with fixed fields $\bar{k}_1 = \bar{k}(\sqrt{u})$, $\bar{k}_2 = \bar{k}(\sqrt{\lambda})$ and $\bar{k}_3 = \bar{k}(\sqrt{u\lambda})$ respectively.

The corresponding $u^0(\overline{D \otimes K})$ are written as $u^0(\overline{D \otimes K}/\overline{k_1})$, $u^0(\overline{D \otimes K}/\overline{k_2})$ and $u^0(\overline{D \otimes K}/\overline{k_3})$.

Corollary 2.9. *Let K/k be a quadratic extension and let ι be the non-trivial automorphism of K/k . Let D be a central division algebra over k with an involution σ of first kind such that $D \otimes_k K$ is division.*

(1) *If $D \otimes K$ is unramified at the discrete valuation of K and K/k is unramified, then $u^0(D \otimes K) = 2u^0(\overline{D \otimes K})$.*

(2) *If $D \otimes K/K$ is ramified, then $u^0(D \otimes K) = u^0(\overline{D \otimes K}/\overline{k_1}) + u^0(\overline{D \otimes K}/\overline{k_3})$.*

(3) *If K/k is ramified, then $u^0(D \otimes K) = u^+(\overline{D}) + u^-(\overline{D})$.*

Proof. (1) Suppose D is unramified and K/k is unramified. Then $\overline{D \otimes K} = \overline{D} \otimes \overline{K}$ and $\overline{K}/\overline{k}$ is a quadratic extension. Let π be a parameter of k . Take $\pi_D = \pi$. Then $\sigma(\pi_D) = \pi_D$ and $\overline{\text{Int}(\pi_D) \circ (\sigma \otimes \iota)} = \overline{\sigma \otimes \iota}$. By Corollary 2.4, $u^0(D \otimes K) = 2u^0(\overline{D \otimes K})$.

(2) Suppose D is ramified. Suppose σ and π_D are as in Lemma 2.5. Then the fixed field of $\overline{\sigma \otimes \iota}$ is $\overline{k_3}$ and the fixed field of $\overline{\text{Int}(\pi_D) \circ (\sigma \otimes \iota)}$ is $\overline{k_1}$ (where $\overline{k_1}$ and $\overline{k_3}$ are as in Remark 2.8). Thus, by Corollary 2.4, we have $u^0(D \otimes K) = u^0(\overline{D \otimes K}/\overline{k_1}) + u^0(\overline{D \otimes K}/\overline{k_3})$.

(3) Suppose K/k is ramified. Let σ_0 be an involution of the first kind on D and $\sigma \simeq \sigma_0 \otimes \gamma$, where γ is the canonical involution of (u, π) . We have $\overline{D \otimes K} = \overline{D}$ and $\overline{\sigma_0 \otimes \iota} = \overline{\sigma}$. Let $\pi_D = \sqrt{\pi} \in K \subset D \otimes K$. Then $\overline{\text{Int}(\pi_D) \circ (\sigma_0 \otimes \iota)} = \overline{\sigma}$. Thus, by Corollary 2.4, $u(D \otimes K, \sigma, \varepsilon) = u(\overline{D}, \overline{\sigma_0}, \varepsilon) + u(\overline{D}, \overline{\sigma_0}, -\varepsilon)$. Hence $u^0(D \otimes K) = u^+(\overline{D}) + u^-(\overline{D})$. \square

We end this section with the following well known

Lemma 2.10. *Let k be a discrete valued field with residue field \overline{k} and completion \widehat{k} . Suppose $\text{char}(\overline{k}) \neq 2$. Let D be a division algebra over k with center K . Let σ be an involution on D such that $K^\sigma = k$. If $D \otimes \widehat{k}$ is division, then $u(D, \sigma, \varepsilon) \geq u(D \otimes \widehat{k}, \widehat{\sigma}, \varepsilon)$, where $\widehat{\sigma} = \sigma \otimes \text{Id}_{\widehat{k}}$.*

Proof. Let v be the discrete valuation on k and $\pi \in k$ be a parameter. Since $D \otimes \widehat{k}$ is division, v extends to a valuation w on D . Let $\varepsilon = \pm 1$ and $\text{Sym}^\varepsilon(D, \sigma) = \{x \in D \mid \sigma(x) = \varepsilon x\}$. Let e_1, \dots, e_r be a k -basis of $\text{Sym}^\varepsilon(D, \sigma)$. Let $a \in \text{Sym}^\varepsilon(D, \sigma) \otimes \widehat{k}$ and write $a = a_1 e_1 + \dots + a_r e_r$ with $a_i \in \widehat{k}$. Let $b_i \in k$ be such that $a_i \equiv b_i$ modulo $\pi^{ew(a)+1}$ and $b = b_1 e_1 + \dots + b_r e_r \in \text{Sym}^\varepsilon(D, \sigma)$, where e is the ramification index $[w(D^*) : v(k^*)]$. Then $w(a) = w(b)$ and $\overline{ab^{-1}} = 1 \in \overline{D \otimes \widehat{k}}$.

Let $s = ab^{-1} \in D \otimes \widehat{k}$. Then $w(s) = 0$ and $\overline{s} = 1$

$$\begin{aligned} a = sb &\implies \widehat{\sigma}(a) = \sigma(b)\widehat{\sigma}(s) &\implies \varepsilon a = \varepsilon b\widehat{\sigma}(s) \\ &\implies a = b\widehat{\sigma}(s) &\implies sb = b\widehat{\sigma}(s) \\ &\implies s = (\text{Int}(b) \circ \widehat{\sigma})(s) &\implies (\text{Int}(b) \circ \widehat{\sigma})|_{k(s)} = \text{Id}_{k(s)}. \end{aligned}$$

Since $k(s)$ is complete, by hensel's lemma, there exists $c \in k(s)$ such that $c^2 = s$ and $\overline{c^2} = \overline{s} = 1 \in \overline{k(s)}$. Then

$$\begin{aligned} (\text{Int}(b) \circ \widehat{\sigma})(c) = c &\implies b\widehat{\sigma}(c) = cb \\ &\implies a = sb = ccb = cb\widehat{\sigma}(c) \\ &\implies \langle a \rangle \simeq \langle b \rangle \otimes \widehat{k} \end{aligned}$$

Let h be an ε -hermitian forms over $(D \otimes \widehat{k}, \sigma)$. Since $D \otimes \widehat{k}$ is division, $h = \langle \alpha_1, \dots, \alpha_n \rangle$ for some $\alpha_i \in \text{Sym}^\varepsilon(D, \sigma) \otimes \widehat{k}$. For each α_i , let $\beta_i \in \text{Sym}^\varepsilon(D, \sigma)$ be such that $\langle \alpha_i \rangle \simeq \langle \beta_i \rangle \otimes \widehat{k}$ and $h_0 = \langle \beta_1, \dots, \beta_n \rangle$. Then h_0 is an ε -hermitian form over (D, σ) and $h_0 \otimes \widehat{k} \simeq h$. If h is anisotropic over \widehat{k} , then h_0 is anisotropic. In particular, $u(D, \sigma, \varepsilon) \geq u(D \otimes \widehat{k}, \sigma \otimes \text{Id}, \varepsilon)$. \square

3. DIVISION ALGEBRAS OVER $\mathcal{A}_i(2)$ -FIELDS

A field k is called an $\mathcal{A}_i(m)$ -field [Lee13, 2.1] if every system of r homogeneous forms of degree m in more than rm^i variables over k has a nontrivial simultaneous zero over a field extension L/k such that $\gcd(m, [L : k]) = 1$.

Let A be a central simple algebra over a field k . We say that A satisfies the *Springer's property* if for any involution σ on A of the first kind, $\varepsilon \in \{1, -1\}$ and for any odd degree extension L/k , if h is an anisotropic ε -hermitian space over (A, σ) , then $h \otimes L$ is anisotropic. In [PSS01], Parimala, Sridharan and Suresh have shown that Springer's property holds for hermitian or skew-hermitian spaces over quaternion algebras with involution of the first kind. In [Wu15], the author has shown that Springer's property holds for hermitian or skew-hermitian spaces over central simple algebras with involution of the first kind over function fields of p -adic curves.

Now we prove Theorem 1.1(i).

Proof. Let σ be an orthogonal involution on D . Let $\text{Sym}^\varepsilon(D, \sigma) = \{x \in D \mid \sigma(x) = \varepsilon x\}$ and $r = \dim_k(\text{Sym}^\varepsilon(D, \sigma))$. Then $r = d(d + \varepsilon)/2$ [KMRT98, 2.6]. Let e_1, \dots, e_r be a k -basis of $\text{Sym}^\varepsilon(D, \sigma)$. Let h be an ε -hermitian form over (D, σ) of rank $n > (1 + \frac{\varepsilon}{d})2^{i-1}$. Then for $x \in D^n$, we have

$$h(x, x) = q_1(x, x)e_1 + \dots + q_r(x, x)e_r,$$

with each q_i a quadratic form over k in d^2n variables [Mah05, proof of prop. 3.6].

Since k is an $\mathcal{A}_i(2)$ -field and $d^2n > d(d + \varepsilon)2^{i-1} = r2^i$, there exists an odd degree extension L/k such that $\{q_1, \dots, q_r\}$ have a simultaneous nontrivial zero over L . Then h_L is isotropic over D_L . By Springer's property, h is isotropic over D . Hence $u(D, \sigma, \varepsilon) \leq (1 + \frac{\varepsilon}{d})2^{i-1}$.

Similarly, if σ is a symplectic involution on D , then $r = d(d - \varepsilon)/2$ and hence $u(D, \sigma, \varepsilon) \leq (1 - \frac{\varepsilon}{d})2^{i-1}$. \square

Next, we prove Theorem 1.1(ii).

Proof. Let σ be an involution on D of the second kind. Let $\text{Sym}(D) = \{x \in D \mid \sigma(x) = x\}$. Then $\text{Sym}(D)$ is vector space over k and $\dim_k \text{Sym}(D) = d^2$, where $d^2 = \dim_K(D)$. Let e_1, \dots, e_{d^2} be a k -basis of $\text{Sym}(D)$. Let h be a hermitian form over (D, σ) of rank $n > 2^{i-1}$. Then, for $x \in D^n$, $h(x, x) \in \text{Sym}(D)$ and we have

$$h(x, x) = q_1(x, x)e_1 + \dots + q_{d^2}(x, x)e_{d^2},$$

with each q_i a quadratic form over k in $2d^2n$ variables.

Since k is an $\mathcal{A}_i(2)$ -field and $2d^2n > 2d^22^{i-1} = d^22^i$, there exists an odd degree extension L/k such that $\{q_1, \dots, q_{d^2}\}$ have a simultaneous nontrivial zero over L . In particular, h_L is isotropic over D_L . By Springer's property, h is isotropic over D . Hence $u^0(D) \leq 2^{i-1}$. \square

Corollary 3.1. *If D is a quaternion division algebra over an $\mathcal{A}_i(2)$ -field k and σ is of the first kind, then $u^+(D) \leq 3 \cdot 2^{i-2}$ and $u^-(D) \leq 2^{i-2}$;*

Proof. Since, by [PSS01, 3.5], (D, σ, ε) satisfies Springer's property, the corollary follows from Theorem 1.1(i). \square

Corollary 3.2. *If D is a quaternion division algebra over a global function field k , then $u^+(D) = 3$, $u^-(D) = 1$, and $u^0(D) = 2$.*

Proof. By Chevalley-Waring theorem [Che35, War35], every finite field is a C_1 -field. By Tsen-Lang theorem [Lan52], every global function field is a C_2 -field. Since every C_2 -field is an $\mathcal{A}_2(2)$ -field [Lee13, between 2.1 and 2.2], by Theorem 1.1(i), $u^+(D) \leq 3$ and $u^-(D) \leq 1$ and by Theorem 1.1(ii), $u^0(D) \leq 2$. The equality follows from Lemma 2.10 and Lemma 2.2. \square

Corollary 3.3. *Let F the function field of an integral variety X over a p -adic field with $p \neq 2$. Let D be a quaternion algebra over F . If $\dim(X) = n$, then $u^+(D) \leq 3 \cdot 2^n$ and $u^-(D) \leq 2^n$.*

Proof. Since D is a quaternion algebra, by [PSS01, 3.5], D satisfies the Springer's property. Since $\dim(X) = n$, by [HB10] and [Lee13], F is a $\mathcal{A}_{n+2}(2)$ -field. Hence the corollary follows from Corollary 3.1. \square

Corollary 3.4. *Let F be a the function field of a p -adic curve. Let D be a division algebra over F with an involution of the first kind.*

- (1) *If D is a quaternion division algebra, then $u^+(D) \leq 6$ and $u^-(D) \leq 2$.*
- (2) *If D is a biquaternion division algebra, then $u^+(D) \leq 5$ and $u^-(D) \leq 3$.*

Proof. By [Sal97, Sal98, 3.4], $\deg(D) = d = 2$ or 4 . If $d = 2$, then D is a quaternion algebra and result follows from Corollary 3.3. Suppose $d = 4$. By [Wu15, 1.5], D satisfies Springer's property. Since F is a $\mathcal{A}_3(2)$ -field, the result follows from Theorem 1.1(i). \square

Corollary 3.5. *Let F the function field of a p -adic curve. Let L/F be a quadratic extension. Let D a division algebra over F with an involution of the first kind. Then $u^0(D \otimes_F L) \leq 4$.*

Proof. By [Wu15, 1.5], D satisfies Springer's property. Since F is a $\mathcal{A}_3(2)$ -field, the result follows Theorem 1.1(ii). \square

4. DIVISION ALGEBRAS OVER SEMI-GLOBAL FIELDS

Let p be an odd prime number. Let F be the function field of a curve over a p -adic field. Let D is a division algebra over F with an involution σ . In this section, we show that the bounds in Corollary 3.4 for the u-invariants of hermitian of forms over central simple algebras over F are in fact exact values. We also compute $u^0(D)$ if D is a quaternion division algebra with an involution of the second kind over F .

Lemma 4.1. *Let k be a complete discrete valued field with residue field \bar{k} . Suppose \bar{k} is a non-archimedean local field or a global function field with $\text{char}(\bar{k}) \neq 2$. Let D be a division algebra over k with an involution of the first kind and K/k a quadratic extension.*

- (1) *If D is a quaternion division algebra, then $u^+(D) = 6$ and $u^-(D) = 2$.*
- (2) *If D is a biquaternion algebra, then $u^+(D) = 5$ and $u^-(D) = 3$.*
- (3) *If $D \otimes_k K$ is a division algebra, then $u^0(D \otimes_k K) = 4$.*

Proof. (1) Suppose D is an unramified quaternion algebra. Then \overline{D} is a quaternion algebra. Since \overline{k} is either a local field or a global function field, by Lemma 2.2 and Corollary 3.2, we have $u^+(\overline{D}) = 3$, $u^-(\overline{D}) = 1$ and $u^-(\overline{D}) = 2$. Thus, by Corollary 2.6(1), $u^+(D) = 2 * 3 = 6$ and $u^-(D) = 2 * 1 = 2$.

Suppose D is a ramified quaternion algebra. Then \overline{D} is a quadratic extension of \overline{k} and by Lemma 2.2 and Corollary 2.6(2), $u^+(D) = 2 + 4 = 6$ and $u^-(D) = 2 + 0 = 2$.

(2) Suppose D is a biquaternion algebra. Since k is a complete discrete valued field with \overline{k} is a global field or local field, D is ramified by a theorem of Albert [Lam05, III, 4.8] and a theorem of Springer [Lam05, VI, 1.9]. Thus \overline{D} is a quaternion algebra and hence by Lemma 2.2 and Corollary 2.6(2), $u^+(D) = 2 + 3 = 5$ and $u^-(D) = 2 + 1 = 3$.

(3) Suppose $D \otimes K$ is a division algebra. Then, by Corollary 2.9, we have either $u^0(D \otimes K) = 2u^0(\overline{D \otimes K})$ or $u^0(D \otimes K) = u^0(\overline{D \otimes K/k_1}) + u^0(\overline{D \otimes K/k_3})$ or $u^0(D \otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0})$ for some central division algebra D_0 unramified over k with $\deg(D) = \deg(D_0)$.

In the case of Corollary 2.9(1), $u^0(D \otimes K) = 2u^0(\overline{D \otimes K}) = 2 * 2 = 4$;

In the case of Corollary 2.9(2), $u^0(D \otimes K) = u^0(\overline{D \otimes K/k_1}) + u^0(\overline{D \otimes K/k_3})$. Since \overline{k} is a p -adic field or a global field, so are $\overline{k_1}$ and $\overline{k_3}$. We have $u(\overline{k_1}) = u(\overline{k_3}) = 4$. Since $\overline{D \otimes K}$ is a quadratic extension of $\overline{k_1}$, we have $u^0(\overline{D \otimes K/k_1}) = \frac{1}{2}u(\overline{k_1}) = 2$. Similarly, $u^0(\overline{D \otimes K/k_3}) = \frac{1}{2}u(\overline{k_3}) = 2$. Thus, we also have $u^0(D \otimes K) = 2 + 2 = 4$.

In the case of Corollary 2.9(3), $u^0(D \otimes K) = u^+(\overline{D_0}) + u^-(\overline{D_0}) = 3 + 1 = 4$. \square

Now we prove our main result Theorem 1.2.

Proof. Since D is a division algebra. By [RS13, 2.6], there exists a divisorial discrete valuation v of F such that $D \otimes F_v$ is division. Since v is a divisorial discrete valuation, the residue field at v is either a p -adic field or a global function field.

(1) and (3) follow from Corollary 3.4, Lemma 4.1(1)(2) and Lemma 2.10.

(2) By [RS13, 2.6], there exists a divisorial discrete valuation v of F such that $D \otimes L \otimes F_v$ is division. Thus, the result follows from Corollary 3.5, Lemma 4.1(3) and Lemma 2.10. \square

5. TENSOR PRODUCT OF QUATERNIONS OVER ARBITRARY FIELDS

In this section, we revisit and prove Theorem 1.3. We begin with the following

Lemma 5.1. *For $n \geq 1$, let $a_n = \frac{4}{5} + \frac{1}{5}(\frac{9}{4})^n$, $b_n = -\frac{1}{5} + \frac{1}{5}(\frac{9}{4})^n$ and $c_n = \frac{1}{5} + \frac{3}{10}(\frac{9}{4})^n$. Then*

$$a_{n+1} = \frac{3}{4}a_n + c_n, \quad b_{n+1} = \frac{3}{2}b_n + \frac{1}{2}c_n, \quad c_n = \frac{1}{2}a_n + b_n, \quad \frac{3}{2}a_n \geq c_n \geq \frac{3}{2}b_n$$

for all $n \geq 1$.

Proof. It follows from definitions of a_n , b_n and c_n above. \square

Now we prove Theorem 1.3.

Proof. By Lemma 2.1, we may assume that $A = H_1 \otimes \cdots \otimes H_n$. Let $\sigma = \tau_1 \otimes \cdots \otimes \tau_n$, where τ_i is the canonical involutions of H_i for $1 \leq i \leq n$. For $n \geq 1$, let $a_n = \frac{4}{5} + \frac{1}{5}(\frac{9}{4})^n$, $b_n = -\frac{1}{5} + \frac{1}{5}(\frac{9}{4})^n$ and $c_n = \frac{1}{5} + \frac{3}{10}(\frac{9}{4})^n$.

We proceed by induction. For $n = 1$, by [Mah05, 3.4] and [Lee84, 2.10] we have $u^+(H_1) \leq a_1 u(k)$, by [Sch85, ch. 10, 1.7], we have $u^-(H_1) \leq b_1 u(k)$ and by [PS13, 4.4], we have $u^0(H_1) \leq c_1 u(k)$.

Suppose $u^+(H_1 \otimes_k \cdots \otimes_k H_n) \leq a_n u(k)$, $u^-(H_1 \otimes_k \cdots \otimes_k H_n) \leq b_n u(k)$ and $u^0(H_1 \otimes_k \cdots \otimes_k H_n) \leq c_n u(k)$.

Let H_1, \dots, H_{n+1} be quaternion algebras over k , τ_i the canonical involution of H_i and $\sigma = \tau_1 \otimes \cdots \otimes \tau_{n+1}$ on $A = H_1 \otimes \cdots \otimes H_{n+1}$. Since H_{n+1} is a quaternion algebra and τ_{n+1} is the canonical involution, there exist $\lambda_{n+1}, \mu_{n+1} \in H_{n+1}^*$ such that $\tau_{n+1}(\lambda_{n+1}) = -\lambda_{n+1}$, $\tau_{n+1}(\mu_{n+1}) = -\mu_{n+1}$, $\lambda_{n+1}\mu_{n+1} = -\mu_{n+1}\lambda_{n+1}$ and $k(\lambda_{n+1})/k$ is a quadratic extension. Let $\lambda = 1 \otimes \cdots \otimes 1 \otimes \lambda_{n+1} \in A$, $\mu = 1 \otimes \cdots \otimes 1 \otimes \mu_{n+1} \in A$ and \tilde{A} be the centralizer of $k(\lambda)$ in A . Then $\tilde{A} = H_1 \otimes \cdots \otimes H_n \otimes k(\lambda)$. Let $\sigma_1 = \sigma|_{\tilde{A}}$ and $\sigma_2 = \text{Int}(\mu^{-1}) \circ \sigma_1$. By [Mah05, 3.1, 3.2], we have σ_1 is unitary, σ_2 and σ are of the same type and

$$u(A, \sigma, \varepsilon) \leq \min\{u(\tilde{A}, \sigma_1, \varepsilon) + \frac{1}{2}u(\tilde{A} \otimes k(\lambda), \sigma_2, -\varepsilon), \frac{1}{2}u(\tilde{A} \otimes k(\lambda), \sigma_1, \varepsilon) + u(\tilde{A} \otimes k(\lambda), \sigma_2, -\varepsilon)\}.$$

Since σ_1 is unitary and $\tilde{A} = H_1 \otimes_k \cdots \otimes_k H_n \otimes k(\lambda)$, by the induction hypothesis, we have $u(\tilde{A}, \sigma_1, \varepsilon) \leq c_n u(k)$. By [PS13, 4.2], $u(\tilde{A}, \sigma_2, -\varepsilon) = u(H_1 \otimes_k \cdots \otimes_k H_n \otimes k(\lambda), \sigma_2, -\varepsilon) \leq \frac{3}{2}u(H_1 \otimes_k \cdots \otimes_k H_n, \tau_1 \otimes \cdots \otimes \tau_n, -\varepsilon)$.

Since both σ and $\tau_1 \otimes \cdots \otimes \tau_n$ are of the first kind and of different types, we have

$$u^+(H_1 \otimes_k \cdots \otimes_k H_{n+1}) \leq \min\{\frac{1}{2}(\frac{3}{2}a_n) + c_n, \frac{3}{2}a_n + \frac{1}{2}c_n\}u(k) = \frac{3}{4}a_n + c_n = a_{n+1}u(k),$$

$$u^-(H_1 \otimes_k \cdots \otimes_k H_{n+1}) \leq \min\{\frac{1}{2}(\frac{3}{2}b_n) + c_n, \frac{3}{2}b_n + \frac{1}{2}c_n\}u(k) = \frac{3}{2}b_n + \frac{1}{2}c_n = b_{n+1}u(k).$$

Finally by [PS13, 4.3], $u^0(H_1 \otimes_k \cdots \otimes_k H_{n+1} \otimes_k K) \leq \min\{\frac{1}{2}a_{n+1} + b_{n+1}, a_{n+1} + \frac{1}{2}b_{n+1}\}u(k) = \frac{1}{2}a_{n+1} + b_{n+1} = c_{n+1}u(k)$. Here Lemma 5.1 was used in all three calculations. \square

Remark. When $n = 2$, $a_2 = \frac{29}{16}$ is the same as that of [PS13, 4.5], $b_2 = \frac{13}{16}$ is smaller than the bound $\frac{17}{16}$ of [PS13, 4.6, 4.7]. When k is a semi-global field, $u^-(D) \leq \lfloor \frac{13}{2} \rfloor = 6$ is smaller than the bound 8 of [PS13, 4.8].

When $n \geq 3$, a_n is smaller than the bound $\frac{3^{2n-6}}{4^n} \cdot 213$ of [PS13, 4.10, 4.11].

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